

A Sub-Supersolution Approach for a Quasilinear Kirchhoff Equation

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Abstract

In this paper we establish an existence result for a quasilinear Kirchhoff equation via a sub and supersolution approach, by using the pseudomonotone operators theory.

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1 Introduction

In this paper we deal with the quasilinear stationary Kirchhoff equation

$$\begin{cases} -M\left(\int_{\Omega} |\nabla u|^2 dx\right) \Delta u = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (P)$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded smooth domain, $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty)$ is a continuous function satisfying

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(f_1) There are a continuous function $h : \overline{\Omega} \times \mathbb{R} \rightarrow [0, +\infty)$ and $\eta \in [0, 2]$ such that $|f(x, t, y)| \leq h(x, t)(1 + |y|^\eta)$ for all $(x, t, y) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$;

and $M : [0, +\infty) \rightarrow [0, +\infty)$ satisfies

(M_1) M is continuous and increasing;

(M_2) There is a positive constant m such that $M(t) \geq m > 0$ for all $t \in \mathbb{R}$.

Problem (P) is a generalization of the classical stationary Kirchhoff equation

$$\begin{cases} -M \left(\int_{\Omega} |\nabla u|^2 dx \right) \Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

As it is well known, problem (1.1) is the general form of the stationary counterpart of the hyperbolic Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \quad (1.2)$$

that appeared at the first time in the work of Kirchhoff [8], in 1883. The equation in (1.2) is called Kirchhoff Equation and it extends the classical D'Alembert wave equation, by considering the effects of the changes in the length of the strings during the vibrations.

The interest of the mathematicians on the so called nonlocal problems like (1.1) (nonlocal because of the presence of the term $M(\|u\|^2)$, which implies that equations in (P) and (1.1) are no longer pointwise equalities) has increased because they represent a variety of relevant physical and engineering situations and requires a nontrivial apparatus to solve them. It is worthwhile to emphasize that the most of the articles on this subject are concerned with the semilinear case, i.e., $f = f(x, u)$.

In several places we should face nonhomogeneous Kirchhoff term, that is, the function M also depends on the variable $x \in \Omega$. For instance, Límaco, Clark and Medeiros [9] attack a biharmonic evolution equation in which the operator is of the form

$$\mathcal{L}u \equiv a(x)u'' + \Delta(b(x)\Delta u) - M\left(x, t, \int_{\Omega} |\nabla u(x, t)|^2 dx\right) \Delta u$$

motivated by the problem of vertical flexion of fully clamped beams. In Figueiredo, Morales-Rodrigo, Santos Junior & Suárez [7] consider a problem whose equation is of the form

$$-M\left(x, \int_{\Omega} |\nabla u|^2 dx\right) \Delta u = f(x, u) \quad \text{in } \Omega,$$

under homogeneous Dirichlet boundary condition, by using a bifurcation argument. Note that for M nonhomogeneous we lose the variational structure and the approach we use in the present article can not be used, at least in a direct way.

In this work, we explore the presence of the gradient term $|\nabla u|$, which makes problem (P) nonvariational, by considering the nonlocal term M with the minimal typical assumptions $(M_1) - (M_2)$ which, up to now, at least to our knowledge, has not been considered yet. We point out that in the original Kirchhoff equation the term M is of the form $M(t) = a + bt$, $a, b > 0$, which enjoys assumptions (M_1) and (M_2) .

Our approach was motivated by Cuesta Leon [1] and in it the method of sub-supersolution and pseudomonotone operator theory play a key role. We should say that here we have to surmount several technical difficulties provoked by the presence of the nonlocal term M .

The method of sub and supersolution for semilinear nonlocal equations has been previously used by some authors. We cite some of them.

In Alves-Corrêa [2] the authors study the problem

$$\begin{cases} -M \left(\int_{\Omega} |\nabla u|^2 dx \right) \Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.3)$$

via sub-supersolution (monotone iteration) by considering $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ nonincreasing and $H(t) = M(t^2)t$ increasing. Note that the typical Kirchhoff term $M(t) = a+bt$, $a, b > 0$ is increasing, i.e., the result in [2] does not include such a M .

In Corrêa [5] the author studies the problem

$$\begin{cases} -a \left(\int_{\Omega} |u|^q dx \right) \Delta u = H(x)f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.4)$$

where $a : \mathbb{R} \rightarrow \mathbb{R}^+$ is a function satisfying $a(s) \geq a_0 > 0 \forall s \in \mathbb{R}$, $s \mapsto s^{\frac{1}{q}}a(s)$ is increasing and $s \mapsto a(s)$ is decreasing. In particular, a is a bounded function. In this work the author uses sub-supersolution combined with fixed point theory.

In Chipot-Corrêa [6] the authors consider the problem

$$\begin{cases} -\mathcal{A}(x, u)\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.5)$$

where, among other things, $\mathcal{A} : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies

$$0 < a_0 \leq \mathcal{A}(x, u) \leq a_{\infty}, \text{ a.e. } x \in \Omega, \forall u \in L^p(\Omega). \quad (1.6)$$

In that work, it is used sub-supersolution via fixed point properties and, again, the nonlocal term is bounded.

Here, we permit, inspired by [1], that the Kirchhoff term M may be of the form of the original one.

Definition 1.1 *We say that $u \in H_0^1(\Omega) \cap L^{\infty}(\Omega)$ is a weak solution of the problem (P) if*

$$M \left(\int_{\Omega} |\nabla u|^2 dx \right) \int_{\Omega} \nabla u \nabla v dx = \int_{\Omega} f(x, u, \nabla u) v dx \quad \forall v \in H_0^1(\Omega). \quad (1.7)$$

The main result of this paper is as follows:

Theorem 1.1 *Assume the hypotheses $(M_1) - (M_2)$ and (f_1) . Moreover, suppose that there are $\bar{u} \in W^{1,\infty}(\Omega)$ and a family $(\underline{u}_\delta) \subset W_0^{1,\infty}(\Omega)$ such that:*

$$\int_{\Omega} \nabla \bar{u} \nabla v dx \geq \int_{\Omega} \frac{1}{m} f(x, \bar{u}, \nabla \bar{u}) v dx \quad \forall v \in H_0^1(\Omega), v \geq 0 \quad \text{and} \quad \bar{u} \geq 0 \quad \text{on} \quad \partial\Omega, \quad (1.8)$$

$$\|\underline{u}_\delta\|_{1,\infty} \rightarrow 0 \quad \text{as} \quad \delta \rightarrow 0,$$

$$\underline{u}_\delta \leq \bar{u} \quad \text{in} \quad \Omega \quad \text{for} \quad \delta \quad \text{small enough},$$

and given $\alpha > 0$, there is $\delta_0 > 0$ such that

$$\int_{\Omega} \nabla \underline{u}_\delta \nabla v dx \leq \int_{\Omega} \frac{1}{\alpha} f(x, \underline{u}_\delta, \nabla \underline{u}_\delta) v dx \quad \forall v \in H_0^1(\Omega), v \geq 0 \quad \text{for} \quad \delta \leq \delta_0. \quad (1.9)$$

Then there is a small enough $\delta > 0$ such that problem (P) has a weak solution u satisfying $\underline{u}_\delta \leq u \leq \bar{u}$.

2 Preliminary Results

In this section we introduce some concepts and results in order to attack problem (P) . The abstract results concerning monotone operators can be found, for instance, in Lions [10], Nečas [11] and Pascali & Sburlan [12]

Definition 2.1 *Let E be a reflexive Banach space and E^* its topological dual. A nonlinear mapping $A : D(A) \subset E \rightarrow E^*$ is said to be monotone if it satisfies*

$$\langle Au - Av, u - v \rangle \geq 0 \quad u, v \in D(A). \quad (2.1)$$

If the inequality (2.1) is strict for $u \neq v$, we say that A is strict monotone. Here, $\langle \cdot, \cdot \rangle$ means the duality pairing between E^* and E .

Definition 2.2 If E is a Hilbert space and $\phi : E \rightarrow \mathbb{R}$ is C^1 -functional, the gradient of ϕ , denoted by $\nabla\phi : E \rightarrow E$, is defined, through the Riesz Representation Theorem, by

$$\langle \nabla\phi(u), w \rangle = \phi'(u)w \quad \forall u, w \in E,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in E .

Lemma 2.1 If E is a Hilbert space and $\phi \in C^1(E, \mathbb{R})$, then ϕ is convex (strictly convex) if, and only if, $\nabla\phi$ is monotone (strictly monotone).

Definition 2.3 Let E be a Banach space and $\mathcal{C} \subset E$ a closed convex set. An operator $T : \mathcal{C} \rightarrow E^*$ is said to be of type (S_+) provided that whenever $x_n \rightharpoonup x$ in E and

$$\limsup_{n \rightarrow +\infty} \langle Tx_n, x_n - x \rangle \leq 0, \quad (2.2)$$

then $x_n \rightarrow x$ in E .

We remark that the condition (2.2) can be rewritten as

$$\limsup_{n \rightarrow +\infty} \langle Tx_n - Tx, x_n - x \rangle \leq 0. \quad (2.3)$$

Definition 2.4 Let E be a Banach space and $B : E \rightarrow E^*$ an operator. We say that B is pseudomonotone if $u_n \rightharpoonup u$ in E and

$$\limsup_{n \rightarrow +\infty} \langle Bu_n, u_n - u \rangle \leq 0, \quad (2.4)$$

then

$$\liminf_{n \rightarrow +\infty} \langle Bu_n, u_n - v \rangle \geq \langle B(u), u - v \rangle \quad \forall v \in E. \quad (2.5)$$

Definition 2.5 We say that $T : E \rightarrow E^*$ is demicontinuous if $x_n \rightarrow x$ in E implies that $Tx_n \rightharpoonup Tx$ in E^* .

Lemma 2.2 Any demicontinuous operator $T : E \rightarrow E^*$ of type (S_+) is pseudomonotone.

Theorem 2.1 *Let E be a reflexive and separable Banach space and $B : E \rightarrow E^*$ an operator satisfying*

(i) *B is coercive, i.e.,*

$$\frac{\langle B(u), u \rangle}{\|u\|} \rightarrow +\infty \text{ as } \|u\| \rightarrow +\infty \quad (2.6)$$

(ii) *B is bounded and continuous;*

(iii) *B is pseudomonotone.*

Then B is surjective, that is, $B(E) = E^$.*

Next, $\|\cdot\|$ will denote the usual norm $\|u\| = \left(\int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2}}$ in $H_0^1(\Omega)$.

Lemma 2.3 *The operator $L : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ given by*

$$\langle Lu, v \rangle = \int_{\Omega} M(\|u\|^2) \nabla u \nabla v dx \quad (2.7)$$

is strictly monotone.

Proof. Let us consider $G : H_0^1(\Omega) \rightarrow \mathbb{R}$ given by

$$G(u) = \frac{1}{2} \widehat{M}(\|u\|^2) \quad \forall u \in H_0^1(\Omega), \quad (2.8)$$

where $\widehat{M}(t) = \int_0^t M(\tau) d\tau$. Because M is positive and continuous, we have that G is strictly convex. Furthermore

$$G'(u)v = \langle \nabla G(u), v \rangle = \int_{\Omega} M(\|u\|^2) \nabla u \nabla v dx = \langle Lu, v \rangle \quad \forall u, v \in H_0^1(\Omega), \quad (2.9)$$

that is, $\nabla G = L$ and so, in view of Lemma 2.1, L is strictly monotone. ■

Lemma 2.4 *L is of type (S_+) .*

Proof. Let (u_n) be a sequence in $H_0^1(\Omega)$ such that

$$u_n \rightharpoonup u \text{ in } H_0^1(\Omega) \quad (2.10)$$

and

$$\limsup_{n \rightarrow +\infty} \langle Lu_n, u_n - u \rangle \leq 0. \quad (2.11)$$

We have to prove that $u_n \rightarrow u$ in $H_0^1(\Omega)$. For this, we first note that

$$\langle Lu_n, u_n - u \rangle = M(\|u_n\|^2) \int_{\Omega} |\nabla u_n|^2 dx - M(\|u_n\|^2) \int_{\Omega} \nabla u_n \nabla u dx$$

that is,

$$\frac{1}{M(\|u_n\|^2)} \langle Lu_n, u_n - u \rangle = \int_{\Omega} |\nabla u_n|^2 dx - \int_{\Omega} \nabla u_n \nabla u dx$$

Note that $M(\|u_n\|^2) \geq m > 0$, and so,

$$0 \geq \limsup_{n \rightarrow \infty} \|u_n\|^2 - \|u\|^2,$$

which implies

$$\|u\|^2 \geq \limsup_{n \rightarrow \infty} \|u_n\|^2 \geq \liminf_{n \rightarrow \infty} \|u_n\|^2 \geq \|u\|^2,$$

from where it follows that $\|u_n\|^2 \rightarrow \|u\|^2$. Invoking the weak convergence $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, we see that $u_n \rightarrow u$ in $H_0^1(\Omega)$, and the proof of the lemma is over. \blacksquare

3 Proof of the Main Theorem

From now on, we fix $R > 0$ large enough such that

$$\|\nabla \bar{u}\|_{\infty}, \|\nabla \underline{u}_{\delta}\|_{\infty} \leq R$$

for all δ small enough, where \bar{u} and u_δ were given in Theorem 1.1. We recall that if $\vec{V} = (V_1, \dots, V_N) \in (L^\infty(\Omega))^N$, we have $\|\vec{V}\|_\infty = \max_{1 \leq i \leq N} \|V_i\|_\infty$. Moreover, we set the function $g_R : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$g_R(t) = \begin{cases} t, & \text{if } |t| \leq R, \\ R, & \text{if } t \geq R, \\ -R, & \text{if } t \leq -R. \end{cases}$$

Here, we would like to point out that

$$g_R(t) = t \quad \text{if } |t| \leq R \quad (3.1)$$

and

$$|g_R(t)| = \min\{R, |t|\} \quad \text{for all } t \in \mathbb{R}.$$

Hence,

$$|g_R(t)| \leq R \quad \text{and} \quad |g_R(t)| \leq |t| \quad \text{for all } t \in \mathbb{R}. \quad (3.2)$$

Taking into account the above function g_R and their properties, we will consider the following auxiliary function $f_R : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty)$ given by

$$f_R(x, t, y) = f(x, t, \vec{g}_R(y)),$$

where $\vec{g}_R(y) = (g_R(y_1), g_R(y_2), \dots, g_R(y_N))$. Using the definition of the function f_R , it follows the ensuing estimates:

$$|f_R(x, t, y)| \leq h(x, t)(1 + |\vec{g}_R(y)|^\eta) \leq h(x, t)(1 + R^\eta N^{\frac{\eta}{2}}) \quad (3.3)$$

and

$$|f_R(x, t, y)| \leq h(x, t)(1 + |\vec{g}_R(y)|^\eta) \leq h(x, t)(1 + |y|^\eta). \quad (3.4)$$

Furthermore, it is crucial observing that

$$f_R(x, t, y) = f(x, t, y) \quad \text{if } |y| \leq R, \quad (3.5)$$

and so,

$$f_R(x, \bar{u}, \nabla \bar{u}) = f(x, \bar{u}, \nabla \bar{u}) \quad \text{and} \quad f_R(x, u_\delta, \nabla \bar{u}_\delta) = f(x, \bar{u}_\delta, \nabla \bar{u}_\delta).$$

Using function f_R , we are able to fix the following auxiliary problem

$$\begin{cases} -M \left(\int_{\Omega} |\nabla u|^2 dx \right) \Delta u &= f_R(x, u, \nabla u) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{cases} \quad (AP)$$

Our intention is proving the existence of a solution u_R for (AP) with $\|\nabla u_R\|_{\infty} \leq R$ if R is large enough and, because of (3.5), we can guarantee that u_R is a solution of the original problem (P).

3.1 Supersolution

In this subsection, we will be concerned on supersolutions of the problem (AP).

Definition 3.1 *We say that $w \in W^{1,\infty}(\Omega)$ is a supersolution of the problem (AP) if*

$$-M(\|w\|^2)\Delta w \geq f_R(x, w, \nabla w) \quad \text{in } \Omega \quad \text{and} \quad w \geq 0 \quad \text{on } \partial\Omega, \quad (3.6)$$

in the weak sense, that is,

$$M(\|w\|^2) \int_{\Omega} \nabla w \nabla v dx \geq \int_{\Omega} f_R(x, w, \nabla w) v dx \quad (3.7)$$

$\forall v \in H_0^1(\Omega)$, $v \geq 0$ a.e. in Ω

How to get a supersolution to the problem (AP)? Under the hypotheses of Theorem 1.1, we know that $\bar{u} \in W^{1,\infty}(\Omega)$ verifies

$$-\Delta \bar{u} \geq \frac{1}{m} f(x, \bar{u}, \nabla \bar{u}) \quad \text{in } \Omega. \quad (3.8)$$

Since $M(t) \geq m > 0$ for all $t \geq 0$ and $f(x, \bar{u}, \nabla \bar{u}) = f_R(x, \bar{u}, \nabla \bar{u})$, we deduce that $\bar{u} \in W^{1,\infty}(\Omega)$ is a supersolution of the problem (AP).

We point out that sub and supersolutions for quasilinear local problems like

$$\begin{cases} -\Delta u &= f(x, u, \nabla u) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega \end{cases} \quad (3.9)$$

were studied in [1].

Lemma 3.1 *Let $u_R \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be a weak solution of the problem (AP) with $0 < u_R \leq \bar{u}$ a.e. in Ω . Then there is a constant $K = K(\|\bar{u}\|_\infty, R)$ such that*

$$\|u_R\|^2 \leq K. \quad (3.10)$$

Proof. Setting $T = \|\bar{u}\|_\infty$, by condition (f_1) combined with (3.3), there is a constant $C = C(T) > 0$ such that

$$|f_R(x, t, y)| \leq C(1 + R^\eta N^{\frac{\eta}{2}}) = C_1 \quad (3.11)$$

for all $(x, t, y) \in \Omega \times [0, T] \times \mathbb{R}^N$. Since u_R is a solution of (AP), we have

$$M(\|u_R\|^2) \int_{\Omega} \nabla u_R \nabla v dx = \int_{\Omega} f_R(x, u_R, \nabla u_R) v dx \quad \forall v \in H_0^1(\Omega) \quad (3.12)$$

and so

$$M(\|u_R\|^2) \|u_R\|^2 = \int_{\Omega} f_R(x, u_R, \nabla u_R) u_R dx. \quad (3.13)$$

Invoking (3.11), we obtain

$$M(\|u_R\|^2) \|u_R\|^2 \leq C_1 \int_{\Omega} |u_R| dx \leq C_1 \int_{\Omega} |\bar{u}| dx \quad (3.14)$$

leading to

$$m \|u_R\|^2 \leq C_2, \quad (3.15)$$

from where it follows that there is $K > 0$ satisfying $\|u_R\|^2 \leq K$. ■

3.2 Subsolution

In this section we will be concerned on subsolutions of (AP) .

Definition 3.2 *We say that $w \in W^{1,\infty}(\Omega)$ is a subsolution of the problem (AP) if*

$$-M(\|w\|^2)\Delta w \leq f_R(x, w, \nabla w) \text{ in } \Omega \text{ and } w \leq 0 \text{ on } \partial\Omega, \quad (3.16)$$

in the weak sense, that is,

$$M(\|w\|^2) \int_{\Omega} \nabla w \nabla v dx \leq \int_{\Omega} f_R(x, w, \nabla w) v dx \quad (3.17)$$

$\forall v \in H_0^1(\Omega)$, $v \geq 0$ a.e. in Ω

In order to construct a subsolution, we consider the family $(\underline{u}_{\delta}) \subset W_0^{1,\infty}(\Omega)$ mentioned in Theorem 1.1, we know that there is $\delta^* > 0$ such that

$$\underline{u}_{\delta} \leq \bar{u} \quad \forall \delta \in [0, \delta^*], \quad (3.18)$$

with

$$\|\underline{u}_{\delta}\|_{1,\infty} \rightarrow 0 \text{ as } \delta \rightarrow 0^+. \quad (3.19)$$

Thereby, fixing $\alpha = \max_{t \in [0,1]} M(t)$, we can reduce if necessary δ^* to get

$$-\Delta \underline{u}_{\delta} \leq \frac{1}{\alpha} f(x, \underline{u}_{\delta}, \nabla \underline{u}_{\delta}) \text{ in } \Omega \text{ and } \underline{u}_{\delta} = 0 \text{ on } \partial\Omega. \quad (3.20)$$

Once that $f(x, \underline{u}_{\delta}, \nabla \underline{u}_{\delta}) = f_R(x, \underline{u}_{\delta}, \nabla \underline{u}_{\delta})$, we can claim that $\underline{u} = \underline{u}_{\delta}$ for $\delta \in (0, \delta^*)$ is a subsolution of (AP) .

3.3 Another Auxiliary Problem

In what follows, we define

$$z_R(x, t, y) = \begin{cases} f_R(x, \underline{u}(x), \nabla \underline{u}(x)), & t \leq \underline{u}(x), \\ f_R(x, t, y), & \underline{u}(x) \leq t \leq \bar{u}(x), \\ f_R(x, \bar{u}(x), \nabla \bar{u}(x)), & t \geq \bar{u}(x) \end{cases}$$

and for $l \in (0, 1)$ we define the function

$$\gamma_R(x, t) = -(\underline{u}(x) - t)_+^l + (t - \bar{u}(x))_+^l.$$

Using the above functions, we consider below a second auxiliary problem

$$\begin{cases} -M \left(\int_{\Omega} |\nabla u|^2 dx \right) \Delta u &= z_R(x, u, \nabla u) - \gamma_R(x, u) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{cases} \quad (3.21)$$

Next, our goal is proving the existence of a solution for the problem (3.21). To this end, we will use Theorem 2.6 to the operator

$$\begin{aligned} B : H_0^1(\Omega) &\rightarrow H^{-1}(\Omega) \\ u &\mapsto B(u) \end{aligned}$$

where

$$\begin{aligned} B(u) : H_0^1(\Omega) &\rightarrow \mathbb{R} \\ v &\mapsto \langle B(u), v \rangle \end{aligned}$$

is given by

$$\langle B(u), v \rangle = M(\|u\|^2) \int_{\Omega} \nabla u \nabla v dx - \int_{\Omega} z_R(x, u, \nabla u) v dx + \int_{\Omega} \gamma_R(x, u) v dx.$$

In what follows, we are going to show that B is onto. So, there exists $u_R \in H_0^1(\Omega)$ such that $B(u_R) = 0$ in $H^{-1}(\Omega)$. Consequently, u_R is a weak solution of the auxiliary problem. If such a solution enjoys $\underline{u} \leq u_R \leq \bar{u}$ a.e. in Ω we get a solution of problem (AP).

Plainly B is continuous. In what follows, we fix our attention to others properties of B in order to apply Theorem 2.6.

Lemma 3.2 *B is coercive.*

Proof. First note that

$$\langle B(u), u \rangle = M(\|u\|^2) \|u\|^2 - \int_{\Omega} z_R(x, u, \nabla u) u dx + \int_{\Omega} \gamma_R(x, u) u dx.$$

It follows from the definition of z_R that there exists $C = C(R) > 0$ such that

$$z_R(x, t, y) \leq C \quad \forall (x, t, y) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$$

and

$$|\gamma(x, t)| \leq C_1 + C_2 t^l \quad \forall (x, t) \in \Omega \times \mathbb{R}.$$

Consequently,

$$|z_R(x, u, \nabla u)u| \leq C|u|$$

and

$$|\gamma(x, u)| \leq C_1|u| + C_2|u|^{l+1}.$$

From these last inequalities,

$$-\int_{\Omega} z_R(x, u, \nabla u)u dx \geq -\int_{\Omega} |z_R(x, u, \nabla u)|u dx \geq -C_1\|u\|$$

and

$$\int_{\Omega} \gamma(x, u)u dx \geq -\int_{\Omega} |\gamma(x, u)u| dx \geq -C_3 \int_{\Omega} |u| dx - C_4 \int_{\Omega} |u|^{l+1} dx,$$

that is,

$$\int_{\Omega} \gamma(x, u)u dx \geq -C_5\|u\| - C_6\|u\|^{l+1}.$$

Since $M(t) \geq m > 0$ for all $t \geq 0$, one has

$$\langle B(u), u \rangle \geq m\|u\|^2 - C_7\|u\| - C_6\|u\|^{l+1}$$

which yields

$$\frac{\langle B(u), u \rangle}{\|u\|} \geq m\|u\| - C_7 - C_6\|u\|^l$$

and the result follows because $l \in (0, 1)$. ■

Lemma 3.3 *B is pseudomonotone.*

Proof. Let $(u_n) \subset H_0^1(\Omega)$ be a sequence satisfying

$$u_n \rightharpoonup u \text{ in } H_0^1(\Omega) \text{ and } \limsup_{n \rightarrow \infty} \langle B(u_n), u_n - u \rangle \leq 0,$$

and recall that

$$\langle B(u_n), u_n - u \rangle = \langle Lu_n, u_n - u \rangle - \int_{\Omega} h(x, u_n, \nabla u_n)(u_n - u) dx + \int_{\Omega} \gamma(x, u_n)(u_n - u) dx. \quad (3.22)$$

Note that

$$\left| \int_{\Omega} z_R(x, u_n, \nabla u_n)(u_n - u) dx \right| \leq C|u_n - u|_1 \rightarrow 0$$

and

$$\int_{\Omega} \gamma_R(x, u_n)|u_n - u| dx \rightarrow 0.$$

Consequently,

$$\limsup_{n \rightarrow \infty} \langle B(u_n), u_n - u \rangle = \limsup_{n \rightarrow \infty} \langle Lu_n, u_n - u \rangle.$$

Since L is an operator of the type (S_+) , it follows that $u_n \rightarrow u$ in $H_0^1(\Omega)$ and invoking the continuity of B , we obtain

$$\liminf_{n \rightarrow \infty} \langle B(u_n), u_n - u \rangle = \langle B(u), u - v \rangle \quad \forall v \in H_0^1(\Omega),$$

showing that B is pseudomonotone. ■

From the above lemmas, the operator B enjoys all the hypotheses of Theorem 2.6 and so B is onto. Consequently, there is $u_R \in H_0^1(\Omega)$ such that $B(u_R) = 0$.

3.4 Existence of Solution for (AP)

As we remarked before, it is enough to show that $\underline{u} \leq u_R \leq \overline{u}$. In this section, we will denote u_R by u .

1st Step. $u \leq \bar{u}$.

For this first step, we take $v = (u - \bar{u})_+$ as a test function. Then,

$$M(\|u\|^2) \int_{\Omega} \nabla u \nabla (u - \bar{u})_+ dx = \int_{\Omega} z_R(x, u, \nabla u) (u - \bar{u})_+ - \int_{\Omega} \gamma_R(x, u) (u - \bar{u})_+$$

Thus,

$$\begin{aligned} \int_{\Omega} \nabla u \nabla (u - \bar{u})_+ dx &= \int_{\Omega} \frac{1}{M(\|u\|^2)} f_R(x, \bar{u}, \nabla \bar{u}) (u - \bar{u})_+ dx - \frac{1}{M(\|u\|^2)} \int_{\Omega} (u - \bar{u})_+^{l+1} dx \\ &\leq \frac{1}{m} \int_{\Omega} f_R(x, \bar{u}, \nabla \bar{u}) (u - \bar{u})_+ dx - \frac{1}{M(\|u\|^2)} \int_{\Omega} (u - \bar{u})_+^{l+1} dx \\ &\leq \int_{\Omega} \nabla \bar{u} \nabla (u - \bar{u})_+ dx - \frac{1}{M(\|u\|^2)} \int_{\Omega} (u - \bar{u})_+^{l+1} dx. \end{aligned}$$

Combining these inequalities, we get

$$0 \leq \int_{\Omega} |\nabla (u - \bar{u})_+|^2 dx \leq -\frac{1}{M(\|u\|^2)} \int_{\Omega} (u - \bar{u})_+^{l+1} dx \leq 0,$$

from where it follows that $u \leq \bar{u}$ in Ω .

2nd Step. $\underline{u} \leq u$.

Firstly, we point out that if $\delta > 0$ is small enough, there is $\beta^* > 0$, independent of δ , such that $\|u\|^2 \leq \beta^*$. Indeed, note that

$$M(\|u\|^2) \int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} z_R(x, u, \nabla u) u dx - \int_{\Omega} \gamma_R(x, u) u dx.$$

By the first step, $\gamma_R(x, u) = -(\underline{u} - u)_+^l$. Then,

$$m\|u\|^2 \leq C \int_{\Omega} |u| dx + \int_{\Omega} (\underline{u} - u)_+^l |u|$$

This last inequality gives

$$m\|u\|^2 \leq C\|u\| + C\|\underline{u}\|_{\infty}^l \|u\| + C\|u\|^{l+1}.$$

Thereby, there is $\beta^* = \beta^*(R, m, l) > 0$, independent of $\delta > 0$ small enough, such that

$$\|u\|^2 \leq \beta^*.$$

In what follows, we reduce $\delta > 0$ if necessary, to get

$$-\Delta \underline{u} \leq \frac{1}{\alpha^*} f_R(x, \underline{u}, \nabla \underline{u})$$

where $\alpha^* = \max_{0 \leq t \leq \beta^*} M(t)$. Choosing $v = (\underline{u} - u)_+$, we obtain

$$\begin{aligned} M(\|u\|^2) \int_{\Omega} \nabla u \nabla (\underline{u} - u)_+ dx &= \int_{\Omega} z_R(x, u, \nabla u) (\underline{u} - u)_+ dx - \int_{\Omega} \gamma_R(x, u) (\underline{u} - u)_+ dx \\ &= \int_{\Omega} z_R(x, \underline{u}, \nabla \underline{u}) (\underline{u} - u)_+ dx + \int_{\Omega} (\underline{u} - u)_+^{l+1} dx \end{aligned}$$

and so

$$\int_{\Omega} \nabla u \nabla (\underline{u} - u)_+ dx = \int_{\Omega} \frac{1}{M(\|u\|^2)} z_R(x, \underline{u}, \nabla \underline{u}) (\underline{u} - u)_+ dx + \int_{\Omega} \frac{1}{M(\|u\|^2)} (\underline{u} - u)_+^{l+1} dx.$$

Hence

$$\begin{aligned} \int_{\Omega} \nabla u \nabla (\underline{u} - u)_+ dx &\geq \int_{\Omega} \frac{1}{\alpha^*} f_R(x, \underline{u}, \nabla \underline{u}) (\underline{u} - u)_+ dx + \frac{1}{M(\|u\|^2)} \int_{\Omega} (\underline{u} - u)_+^{l+1} dx \\ &\geq \int_{\Omega} \nabla \underline{u} \nabla (\underline{u} - u)_+ dx + \frac{1}{M(\|u\|^2)} \int_{\Omega} (\underline{u} - u)_+^{l+1} dx. \end{aligned}$$

Then

$$0 \geq \int_{\Omega} |\nabla (\underline{u} - u)_+|^2 dx + \frac{1}{M(\|u\|^2)} \int_{\Omega} (\underline{u} - u)_+^{l+1} dx \geq 0$$

and this implies that $(\underline{u} - u)_+ = 0$. Thus, $\underline{u} \leq u$ in Ω , and the proof of the existence of solution for (AP) is over.

3.5 Existence of Solution for (P)

To begin with, we observe that in the last subsection we proved the existence of a solution u_R of (AP) verifying $\underline{u} \leq u_R \leq \bar{u}$ in Ω . Here, we would like point out that \underline{u} and \bar{u} does not depend of R , for R large enough. In what follows, we denote u_R by u .

Our goal is to show that there is $R^* > 0$ such that

$$\|\nabla u\|_\infty \leq R \quad \text{for } R \geq R^*.$$

By Elliptic Regularity,

$$u \in W^{2,p}(\Omega) \quad \forall p \in [1, +\infty),$$

because $f_R \in L^\infty([0, +\infty))$ and $u \in L^\infty(\Omega)$. From now on, we will fix p such that

$$W^{2,p}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega}) \quad (3.23)$$

is a continuous embedding. Now, we observe that u is a solution of the problem

$$-\Delta u + u = B_R(x)(1 + |\nabla u|^2),$$

where

$$B_R(x) = \frac{u + \frac{f_R(x, u(x), \nabla u(x))}{M(\|u\|^2)}}{1 + |\nabla u|^2}.$$

Once that

$$|f_R(x, t, y)| \leq h(x, t)(1 + |y|^\eta) \quad \forall (x, t, y) \in \Omega \times \mathbb{R} \times \mathbb{R}^N,$$

combining the fact that $\eta \in [0, 2]$, $\underline{u} \leq u \leq \bar{u}$ in Ω and $\|\underline{u}\|_\infty, \|\bar{u}\|_\infty$ does not depend of R , for R large enough, the conditions (f_1) and (M_2) guarantee the existence of $C^* > 0$, independent of R , such that

$$|B_R(x)| \leq C^* \quad \forall x \in \Omega, \quad \text{for } R \text{ large enough.}$$

Thereby, there is $R_1 > 0$ such that

$$\|B_R\|_\infty \leq C^* \quad \forall R > R_1. \quad (3.24)$$

By using a result due to Amann & Crandall [3, Lemma 4], there is an increasing function $\gamma_0 : [0, +\infty) \rightarrow [0, \infty)$, depending only of Ω , p and N , and satisfying

$$\|u\|_{W^{2,p}(\Omega)} \leq \gamma_0(\|B_R\|_\infty).$$

Combining the last inequality with (3.23) and (3.24), we get

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leq C\gamma_0(C^*),$$

for some $C > 0$. Fixing

$$K_1 = C\gamma_0(C^*),$$

we derive that

$$\left| \frac{\partial u(x)}{\partial x_i} \right| \leq K_1 \quad \forall x \in \overline{\Omega} \quad \text{and } i = 1, 2, \dots, N.$$

Thereby,

$$|\nabla u(x)| \leq NK_1 \quad \forall x \in \overline{\Omega},$$

implying that

$$\max_{x \in \overline{\Omega}} |\nabla u(x)| \leq NK_1.$$

Fixing $R_2 = NK_1$ and $R \geq R^* = \max\{R_1, R_2\}$, it follows that

$$\max_{x \in \overline{\Omega}} |\nabla u(x)| \leq R,$$

showing that u is a solution of (P) if $R \geq R^*$.

4 Applications

In this section, we will present two situations in which our main theorem works.

Application 1: Our first application is the following problem

$$\begin{cases} -M \left(\int_{\Omega} |\nabla u|^2 dx \right) \Delta u = \lambda |u|^q + |u|^p + \mu |\nabla u|^q & \text{in } \Omega, \\ u(x) > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.1)$$

where λ is a positive parameter, $0 < q < 1 < p < +\infty$ and M verifies conditions $(M_1) - (M_2)$.

Here, we must observe that the above problem is a nonlocal version of a well known result due to Ambrosetti, Brezis & Cerami [4] with an additional gradient term $|\nabla u|^q$.

We begin observing that it is easy to find a positive function \bar{u} verifying the inequality

$$-\Delta \bar{u} \geq \frac{1}{m}(\lambda \bar{u}^q + \bar{u}^p + \mu |\nabla \bar{u}|^q)$$

if λ, μ are small enough. It is enough to follow the ideas found in Ambrosetti, Brezis & Cerami [4]. Indeed, let $0 < e$ in Ω , $e \in C^1(\bar{\Omega})$ the only solution of

$$\begin{cases} -\Delta e = 1 & \text{in } \Omega, \\ e = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.2)$$

We now take $S > 0$ such that

$$m \geq \frac{1}{S^{1-q}}(\lambda \|e\|_\infty^q + \mu \|\nabla e\|_\infty^q) + S^{p-1} \|e\|_\infty^p. \quad (4.3)$$

A straightforward computation shows that there is $0 < \lambda^*$ such that for $0 < \lambda, \mu < \lambda^*$ there is $S > 0$ such that the inequality (4.3) holds true. Hence we can take $\bar{u} := Se \in W^{1,\infty}(\Omega)$, S as above, so that the first inequality in the Theorem 1.1 is satisfied.

Now, fixed $\lambda, \mu > 0$ as before, we consider the family (u_δ) with $u_\delta = \delta \varphi_1$, φ_1 is a positive eigenfunction associated with the principal eigenvalue λ_1 of $(-\Delta, H_0^1(\Omega))$. A simple computation also gives for all $\alpha > 0$ fixed, there exist $\delta^* > 0$ such that

$$-\Delta u_\delta \leq \frac{1}{\alpha}(\lambda u_\delta^q + u_\delta^p + \mu |\nabla u_\delta|^q) \quad \text{in } \Omega.$$

As it is well known, we can consider $\delta > 0$ sufficiently small such that $u_\delta \leq \bar{u}$.

From the above commentaries, we can apply Theorem 1.1 to prove the existence of a weak solution u for (4.1) satisfying $u_\delta \leq u \leq \bar{u}$.

Application 2: Our next application is concerning the problem

$$\begin{cases} -M \left(\int_{\Omega} |\nabla u|^2 dx \right) \Delta u = Au^q(B-u) + |\nabla u|^\eta & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (4.4)$$

where A, B are positive constants satisfying some properties which will be established later, $\eta \in (1, 2]$ and $q \in (0, 1)$. We will find a solution u satisfying $0 < u \leq B$ in Ω . First of all, let us consider the continuous function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, +\infty)$ defined as

$$f(x, t, y) = \begin{cases} |y|^\eta, & \text{if } t \geq B \\ At^q(B-t) + |y|^\eta, & \text{if } 0 \leq t \leq B \\ |y|^\eta, & \text{if } t \leq 0. \end{cases}$$

It is clear that the function $\bar{u} \equiv B$ belongs to $W^{1,\infty}(\Omega)$ and satisfies the assumption (3.7) in the Theorem 1.1.

If λ_1 is the principal eigenvalue of $(-\Delta, H_0^1(\Omega))$ associated to the eigenfunction $\varphi_1 > 0$ in Ω , for each $\alpha > 0$, there is $\delta^* > 0$ such that

$$\lambda_1 \delta \varphi_1 \leq \frac{A}{\alpha} (\delta \varphi_1)^q (B - \delta \varphi_1) + \frac{1}{\alpha} |\nabla(\delta \varphi_1)|^\eta \quad \forall \delta \in (0, \delta^*].$$

Taking $\underline{u}_\delta := \delta \varphi_1$ we get $\|\underline{u}_\delta\|_{1,\infty} \rightarrow 0$ as $\delta \rightarrow 0$, $\underline{u}_\delta \leq \bar{u} \equiv B$ in Ω , if $\delta > 0$ is small enough and a straightforward calculation shows that the inequality (3.17) holds true. Hence, for δ sufficiently small, problem (P) possesses a weak solution u satisfying $\underline{u}_\delta \leq u \leq B$. Consequently, such a function is a solution of the problem (4.4).

Remark 4.1 *For some applications concerning the quasilinear problem (P), with $M \equiv 1$, still using a sub and supersolution approach, the reader may consult Xavier [13] and the references therein.*

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